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Abstract. A popular way to study the tail of a distribution is to consider its extreme quantiles. While this is a standard procedure for univariate distributions, it is harder for multivariate ones, primarily because there is no universally accepted definition of what a multivariate quantile should be. In this paper, we focus on extreme geometric quantiles. Their asymptotics are established, both in direction and magnitude, under suitable moment conditions, when the norm of the associated index vector tends to one. In particular, it appears that if a random vector has a finite covariance matrix, then the magnitude of its extreme geometric quantiles grows at a fixed rate. We take advantage of these results to define an estimator of extreme geometric quantiles of such a random vector. The consistency and asymptotic normality of the estimator are established and our results are illustrated on some numerical examples.

AMS Subject Classifications: 62H05, 62G20, 62G32.

Keywords: Extreme quantile, geometric quantile, consistency, asymptotic normality.

1 Introduction

Let X be a random vector in $\mathbb{R}^d$. Up to now, several definitions of multivariate quantiles of X have been proposed in the statistical literature. We refer to Serfling (2002) for a review of various possibilities for this notion. Here, we focus on the notion of "spatial" or "geometric" quantiles, introduced by Chaudhuri (1996), which generalises the characterisation of a univariate quantile shown in Koenker and Bassett (1978). For a given vector u belonging to the unit open ball B^d of $\mathbb{R}^d$, where $d \geq 2$, a geometric quantile with index vector u is any solution of the optimisation problem defined by

$$\arg\min_{q \in \mathbb{R}^d} \mathbb{E}(\phi(u, X - q) - \phi(u, X)), \tag{1}$$

with the loss function $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R},\ (u,t) \mapsto \|t\| + \langle u, t\rangle$, where $\langle \cdot, \cdot \rangle$ is the usual scalar product on $\mathbb{R}^d$ and $\|\cdot\|$ is the associated Euclidean norm. Note that $q(u) \in \mathbb{R}^d$ possesses both a direction and magnitude. It can be seen that geometric quantiles are in fact special cases of $M-$quantiles introduced by Breckling and Chambers (1988) which were further analysed by Koltchinskii (1997). Besides, such quantiles have various strong properties. First, the quantile with index vector $u \in B^d$ is unique whenever the distribution of X is not concentrated on a single straight line in $\mathbb{R}^d$ (see Chaudhuri, 1996, or Theorem 2.17 in Kemperman, 1987). Second, although they are not fully affine equivariant, they are equivariant under any orthogonal transformation (Chaudhuri, 1996). Third, geometric quantiles characterise the associated distribution. Namely, if two random variables X and Y yield the same quantile function q, then X and Y have the same distribution (Koltchinskii, 1997). Finally, for $u = 0$, the well-known L^2-geometric median is obtained, which is the simplest example of a "central" quantile (see Small, 1990). We point out that one may compute an estimation of the geometric median in an efficient way, see Cardot *et al.* (2013).

These properties make geometric quantiles reasonable candidates when trying to define multivariate quantiles, which is why their estimation was studied in several papers. We refer for instance to Chaudhuri (1996), who established a Bahadur expansion for the estimator of geometric quantiles obtained by solving the sample counterpart of problem (1). Chakraborty (2001) then introduced a transformation-retransformation procedure to obtain affine equivariant estimates of multivariate quantiles. This notion was extended to a multiresponse linear model by Chakraborty (2003). Recently, Dhar *et al.* (2014) defined a multivariate quantile-quantile plot using geometric quantiles. Conditional geometric quantiles can also be defined by substituting a conditional expectation to the expectation in (1). We refer to Cadre and Gannoun (2000) for the estimation of the conditional geometric median and to Cheng and de Gooijer (2007) for the estimation of an arbitrary conditional geometric quantile. The estimation of a conditional median when there is an infinite-dimensional covariate is considered in Chaouch and Laïb (2013).

Our focus in this paper is rather on extreme geometric quantiles, obtained when $\|u\| \to 1$. The theory of univariate extreme quantiles is well established, see for instance the monograph by de Haan and Ferreira (2006). On the contrary, the few works on extreme multivariate quantiles rely on the study of extreme level sets of the probability density function of X when it is absolutely continuous with respect to the Lebesgue measure. We refer for instance to Cai *et al.* (2011) for an application to the estimation of extreme risk regions for financial data or to Einmahl *et al.* (2013) who focus on the case of bivariate distributions with an application to insurance data. One can also analyse extreme quantiles of multivariate datasets by selecting a univariate variable and considering the other variables as covariates. This amounts to estimating conditional univariate extreme quantiles: for a finite-dimensional covariate, this problem is considered in Daouia *et al.* (2013), the case of a functional covariate being addressed in Gardes and Girard (2012).

In this study, we provide an equivalent of the direction and magnitude of the extreme geometric quantile

$q(u)$, $\|u\| \to 1$ under suitable moment conditions. A particular corollary of our results is that the magnitude of the extreme geometric quantiles of a random vector X having a finite covariance matrix grows at a fixed rate. Moreover, in this case, the magnitude of the extreme geometric quantiles is asymptotically characterised by the covariance matrix of X. This property opens the door to the definition of an extreme quantile estimator, whose asymptotic properties are studied in this work.

The outline of the paper is as follows. Asymptotic properties of geometric quantiles are stated in Section 2. An application to the estimation of extreme geometric quantiles is given in Section 3. Some examples and illustrations of our results are presented in Section 4. Section 5 offers a couple of concluding remarks. Proofs are deferred to Section 6.

2 Asymptotic behaviour of extreme geometric quantiles

From now on, we assume that the distribution of X is not concentrated on a single straight line in $\mathbb{R}^d$ and non-atomic. We shall reformulate the optimisation problem (1) as

$$\arg\min_{q\in\mathbb{R}^d} \psi(u, q)$$

where $\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R},\ (u, q) \mapsto \mathbb{E}(\phi(u, X - q) - \phi(u, X))$ can be rewritten as

$$\psi(u, q) = \mathbb{E}(\|X - q\| - \|X\|) - \langle u, q\rangle. \tag{2}$$

Chaudhuri (1996) proved that in this context, the solution $q(u)$ of (1), namely the geometric quantile with index vector u, exists and is unique for every $u \in B^d$. Define further that $t/\|t\| = 0$ if $t = 0$; if $u \in \mathbb{R}^d$ is such that there is a solution $q(u) \in \mathbb{R}^d$ to problem (1), then the gradient of $q \mapsto \psi(u, q)$ must be zero at $q(u)$, that is

$$u + \mathbb{E}\left(\frac{X - q(u)}{\|X - q(u)\|}\right) = 0. \tag{3}$$

This condition immediately entails that if $u \in \mathbb{R}^d$ is such that problem (1) has a solution $q(u)$, then $\|u\| \le 1$. In fact, we can prove a stronger result:

Proposition 1. *The optimisation problem (1) has a solution if and only if $u \in B^d$.*

Moreover, remarking that the function $\psi(u, \cdot)$ is strictly convex, Chaudhuri (1996) proved the following characterisation of a geometric quantile: for every $u \in B^d$, $q(u)$ is the solution of problem (1) if and only if it satisfies equation (3). In particular, this entails that the function $G : \mathbb{R}^d \to B^d$ defined by

$$\forall q \in \mathbb{R}^d,\ G(q) = -\mathbb{E}\left(\frac{X - q}{\|X - q\|}\right)$$

is a continuous bijection. Proposition 2.6(iii) in Koltchinskii (1997) shows that the inverse of the function G, *i.e* the geometric quantile function $u \mapsto q(u)$, is also continuous on B^d.

In most cases however, computing explicitly the function G is a hopeless task, which makes it impossible to obtain a closed-form expression for the geometric quantile function. It is thus of interest to prove general results about the geometric quantile $q(u)$, especially regarding its direction and magnitude. Our first main result focuses on the special case of spherically symmetric distributions.

Proposition 2. *If X has a spherically symmetric distribution then:*

(i) The map $u \mapsto q(u)$ commutes with every linear isometry of $\mathbb{R}^d$. Especially, the norm of a geometric quantile $q(u)$ only depends on the norm of u.

(ii) For all $u \in B^d$, the geometric quantile $q(u)$ has direction u if $u \neq 0$ and $q(0) = 0$ otherwise.

(iii) The function $\|u\| \mapsto \|q(u)\|$ is a continuous increasing function on $[0, 1)$.

(iv) It holds that $\|q(u)\| \to \infty$ as $\|u\| \to 1$.

Although the first and third statement of Proposition 2 cannot be expected to hold true for a random variable which is not spherically symmetric, one may wonder if the second and fourth statement, namely that a geometric quantile shares the direction of its index vector and that the norm of the geometric quantile function tends to infinity on the unit sphere, can be extended to the general case. The next result, which examines the behaviour of the geometric quantile function near the boundary of the open ball B^d, provides an answer to this question.

Theorem 1. *Let S^{d-1} be the unit sphere of $\mathbb{R}^d$.*

(i) It holds that $\|q(v)\| \to \infty$ as $\|v\| \to 1$.

(ii) Moreover, if $v \to u$ with $u \in S^{d-1}$ and $v \in B^d$ then $q(v)/\|q(v)\| \to u$.

Theorem 1 shows two properties of geometric quantiles: first, the norm of the geometric quantile $q(v)$ with index vector v diverges to infinity as $\|v\| \uparrow 1$. In other words, Proposition 2(iv) still holds for any distribution. This is a rather intriguing property of geometric quantiles, since it holds even if the distribution of X has a compact support. A related point is the fact that sample geometric quantiles do not necessarily lie within the convex hull of the sample, see Breckling *et al.* (2001) for a counter-example. Second, if $v \to u \in S^{d-1}$ then the geometric quantile $q(v)$ has asymptotic direction u. Proposition 2(ii) thus remains true asymptotically for any distribution.

It is possible to specify the convergences obtained in Theorem 1 under moment assumptions. Theorem 2 provides a first-order expansion of the direction and of the magnitude of an extreme geometric quantile $q(\alpha u)$ in the direction u, where u is a unit vector and α tends to 1.

Theorem 2. *Let $u \in S^{d-1}$.*

(i) If $\mathbb{E}\|X\| < \infty$ then $q(\alpha u) - \{\|q(\alpha u)\| u + \mathbb{E}(X - \langle X, u\rangle u)\} \to 0$ as $\alpha \uparrow 1$.

(ii) If $\mathbb{E}\|X\|^2 < \infty$ and Σ denotes the covariance matrix of X then

$$\|q(\alpha u)\|^2(1-\alpha) \to \frac{1}{2}\left(\operatorname{tr}\Sigma - u'\Sigma u\right) > 0 \quad \text{as} \quad \alpha \uparrow 1.$$

As a consequence of Theorem 2, it appears that if X has a finite covariance matrix Σ then the magnitude of an extreme geometric quantile in the direction u is determined (in the asymptotic sense) by Σ. In other words, since the asymptotic direction of an extreme geometric quantile in the direction u is exactly u by Theorem 1, it follows that the extreme geometric quantiles of two probability distributions which admit the same finite covariance matrix are asymptotically equivalent. Furthermore, we observe that

$$\frac{\|q(\beta u)\|}{\|q(\alpha u)\|} = \left(\frac{1-\alpha}{1-\beta}\right)^{1/2} (1 + \mathrm{o}(1))$$

when $\alpha \to 1$ and $\beta \to 1$. In other words, given an arbitrary extreme geometric quantile, one can deduce the asymptotic behaviour of every other extreme geometric quantile sharing its direction, independently of the distribution. This is fundamentally different from the univariate case when deducing the value of an extreme quantile from another one requires the knowledge of the extreme-value index of the distribution, see de Haan and Ferreira (2006), Chapter 4. Our results can actually be used to define a consistent and asymptotically Gaussian estimator of extreme geometric quantiles, as shown in Section 3 below.

3 An estimator of extreme geometric quantiles

Let $X_1, \ldots, X_n$ be independent random copies of a random vector X having a finite covariance matrix Σ. It follows from Theorem 2 that any extreme geometric quantile $q(\alpha u)$ of X, with $\alpha \uparrow 1$ and $u \in S^{d-1}$ can be approximated by:

$$q_{\mathrm{eq}}(\alpha u) := (1-\alpha)^{-1/2} \left[\frac{1}{2}\left(\operatorname{tr}\Sigma - u'\Sigma u\right)\right]^{1/2} u. \tag{4}$$

This can be used to define an estimator of the extreme geometric quantiles of X: let $\overline{X}_n = n^{-1}\sum_{k=1}^{n} X_k$ be the sample mean and

$$\widehat{\Sigma}_n = \frac{1}{n}\sum_{k=1}^{n}(X_k - \overline{X}_n)(X_k - \overline{X}_n)'$$

be the empirical estimator of the covariance matrix Σ of X. Let further (α_n) be an increasing sequence of positive real numbers tending to 1. Our estimator $\widehat{q}_n(\alpha_n u)$ of $q(\alpha_n u)$ is then

$$\widehat{q}_n(\alpha_n u) = (1-\alpha_n)^{-1/2} \left[\frac{1}{2}\left(\operatorname{tr}\widehat{\Sigma}_n - u'\widehat{\Sigma}_n u\right)\right]^{1/2} u.$$

The consistency of $\widehat{q}_n(\alpha_n u)$ is examined in the next result.

Theorem 3. *Let $u \in S^{d-1}$ and assume that $\alpha_n \uparrow 1$. If $\mathbb{E}\|X\|^2 < \infty$ then*

$$\sqrt{1-\alpha_n}\,(\widehat{q}_n(\alpha_n u) - q(\alpha_n u)) \to 0 \quad \textit{almost surely as} \quad n \to \infty.$$

This result actually means that the extreme geometric quantile estimator is relatively consistent in the sense that

$$\frac{\widehat{q}_n(\alpha_n u) - q(\alpha_n u)}{\|q(\alpha_n u)\|} \to 0 \ \text{ almost surely as } \ n \to \infty,$$

since $\|q(\alpha_n u)\|^{-1} = \mathrm{O}(\sqrt{1-\alpha_n})$, see Theorem 2(ii). This normalisation could be expected since the quantity to be estimated diverges in magnitude. Under the additional assumption that X has a finite fourth moment, an asymptotic normality result can be established for this estimator:

Theorem 4. *Let $u \in S^{d-1}$ and assume that $\alpha_n \uparrow 1$ is such that $n(1-\alpha_n) \to 0$. If $\mathbb{E}\|X\|^4 < \infty$ then*

$$\sqrt{n(1-\alpha_n)}\,(\widehat{q}_n(\alpha_n u) - q(\alpha_n u)) \xrightarrow{d} Z \quad as \quad n \to \infty$$

where Z is a Gaussian centred random vector.

Let us highlight that the covariance matrix of the Gaussian limit in Theorem 4 essentially depends on the covariance matrix M of the Gaussian limit of $\sqrt{n}(\widehat{\Sigma}_n - \Sigma)$, see the proof in Section 6. Although the matrix M has a heavy and complicated expression (see *e.g.* Neudecker and Wesselman, 1990), it can be estimated when $\mathbb{E}\|X\|^4 < \infty$, which makes it possible to construct asymptotic confidence regions for extreme geometric quantiles.

Extreme geometric quantiles can thus be consistently estimated by $\widehat{q}_n(\alpha_n u)$, whatever the "order" α_n, and an asymptotic normality result is obtained when $\alpha_n \uparrow 1$ quickly enough. The proposed estimator is thus able to extrapolate arbitrarily far from the original sample. This is very different from the univariate case, where the empirical quantile $\widehat{q}_n(\alpha_n) = \inf\{t \in \mathbb{R} \,|\, \widehat{F}(t) \geq \alpha_n\}$, deduced from the empirical cumulative distribution function $\widehat{F}$, estimates the true quantile $q(\alpha_n)$ consistently only if α_n converges to 1 slowly enough. The extrapolation with faster rates α_n is then handled assuming that the underlying distribution function is heavy-tailed and by using adapted estimators, see *e.g.* Weissman (1978) and the monograph by de Haan and Ferreira (2006).

4 Numerical illustrations

In this section, our main results are illustrated, particularly Theorems 2, 3 and 4 in the bivariate case $d = 2$ to make the display easier. In this framework, $u \in S^1$ can be represented by an angle and we may write $u = u_\theta = (\cos\theta, \sin\theta)$, $\theta \in [0, 2\pi)$. The iso-quantile curves $\mathcal{C}q(\alpha) = \{q(\alpha u_\theta),\ \theta \in [0, 2\pi)\}$ and their estimates $\mathcal{C}\widehat{q}_n(\alpha) = \{\widehat{q}_n(\alpha u_\theta),\ \theta \in [0, 2\pi)\}$ can then be considered in order to get a grasp of the behaviour of extreme quantiles in every direction. The following two distributions are considered for the random vector X:

- the centred Gaussian multivariate distribution $\mathcal{N}(0, v_X, v_Y, v_{XY})$, with probability density function:

$$\forall x, y \in \mathbb{R},\ f(x,y) = \frac{1}{2\pi\sqrt{\det\Sigma}} \exp\left(-\frac{1}{2}\begin{pmatrix} x \\ y \end{pmatrix}' \Sigma^{-1} \begin{pmatrix} x \\ y \end{pmatrix}\right) \quad \text{with } \Sigma = \begin{pmatrix} v_X & v_{XY} \\ v_{XY} & v_Y \end{pmatrix}.$$

- a double exponential distribution $\mathcal{E}(\lambda_-, \mu_-, \lambda_+, \mu_+)$, with λ_-, μ_-, λ_+, $\mu_+ > 0$, whose probability density function is:

$$\forall x,y\in\mathbb{R},\ f(x,y)=\begin{cases}\dfrac{\lambda_+\mu_+}{4}e^{-\lambda_+|x|-\mu_+|y|} & \text{if } \ xy>0,\\[2ex] \dfrac{\lambda_-\mu_-}{4}e^{-\lambda_-|x|-\mu_-|y|} & \text{if } \ xy\le 0.\end{cases}$$

In this case, X is centred and has covariance matrix

$$\Sigma=\begin{pmatrix}\dfrac{1}{\lambda_-^2}+\dfrac{1}{\lambda_+^2} & \dfrac{1}{2}\left[\dfrac{1}{\lambda_+\mu_+}-\dfrac{1}{\lambda_-\mu_-}\right]\\[2ex] \dfrac{1}{2}\left[\dfrac{1}{\lambda_+\mu_+}-\dfrac{1}{\lambda_-\mu_-}\right] & \dfrac{1}{\mu_-^2}+\dfrac{1}{\mu_+^2}\end{pmatrix}.$$

In our study, three different sets of parameters were used for each distribution, in order that the related covariance matrices coincide:

- $\mathcal{N}(0,1/2,1/2,0)$ and $\mathcal{E}(2,2,2,2)$ with spherical covariance matrices;
- $\mathcal{N}(0,1/8,3/4,0)$ and $\mathcal{E}(4,2\sqrt{2/3},4,2\sqrt{2/3})$ with diagonal covariance matrices;
- $\mathcal{N}(0,1/2,1/2,1/6)$ and $\mathcal{E}(2\sqrt{3},2\sqrt{3},2\sqrt{3/5},2\sqrt{3/5})$ with full covariance matrices.

In each case, we carry out the following computations:

- for each $\alpha\in\{0.99,\ 0.995,\ 0.999\}$, the true quantile curves $\mathcal{C}q(\alpha)$ obtained by solving problem (1) numerically, as well as their analogues $\mathcal{C}q_{\mathrm{eq}}(\alpha)$ using approximation (4) are computed. The normalised squared approximation error

  $$e(\alpha)=(1-\alpha)\int_0^{2\pi}\|q_{\mathrm{eq}}(\alpha u_\theta)-q(\alpha u_\theta)\|^2\,d\theta$$

  is then recorded.

- for each value of α, we draw $N=1000$ replications of an $n-$sample $(X_1,\ldots,X_n)$ of independent copies of X, with $n\in\{100,200,500\}$. The estimated quantile curves $\mathcal{C}\widehat{q}_n^{(j)}(\alpha)$ corresponding to the $j-$th replication and the associated normalised squared error

  $$E_n^{(j)}(\alpha)=(1-\alpha)\int_0^{2\pi}\left\|\widehat{q}_n^{(j)}(\alpha u_\theta)-q(\alpha u_\theta)\right\|^2\,d\theta$$

  are computed as well as the mean squared error $E_n(\alpha)=N^{-1}\sum_{j=1}^N E_n^{(j)}(\alpha)$.

The true quantile curves, as well as the approximated and the estimated ones are displayed on Figures 1–6 in the case $n=200$ and $\alpha=0.995$. The true quantile curves look very similar on Figures 1 and 4, on Figures 2 and 5 and Figures 3 and 6. This is in accordance with Theorem 2: eventually, extreme geometric quantiles only depend on the covariance matrix of the underlying distribution. Moreover, the approximated quantiles curves are close to the true ones in all cases, and the estimated quantile curves are satisfying in all situations with a moderate variability. Similar results were observed for $n=100,500$

and $\alpha = 0.99, 0.999$. We do not report the graphs here for the sake of brevity; we do however display the approximation and estimation errors in Table 1. Unsurprisingly, the estimation error $E_n(\alpha)$ decreases as the sample size n increases. Both approximation and estimation errors $e(\alpha)$ and $E_n(\alpha)$ have a stable behaviour with respect to α.

5 Concluding remarks

In this paper, we established the asymptotics, both in direction and magnitude, of extreme geometric quantiles. A particular consequence of our results is that if the underlying distribution possesses a finite covariance matrix Σ, then an extreme geometric quantile may be estimated accurately, no matter how extreme it is, with the help of the standard empirical estimator of Σ. This is supported by our numerical results.

This work, however, was carried out under moment conditions such as the existence of finite first and second-order moments for $\|X\|$. It would definitely be interesting to see if our conclusions carry over, to some extent, to the case when these assumptions are violated. Furthermore, although geometric quantiles make an appealing candidate for multivariate quantiles, they lack a couple of nice properties such as affine equivariance, for instance. To tackle this issue, one may apply a transformation-retransformation procedure, see Serfling (2010); such procedures admit sample analogues, see for instance Chakraborty *et al.* (1998) and Chakraborty (2001). Future work on extreme geometric quantiles thus includes building and studying an analogue of our estimator for transformed-retransformed data.

6 Proofs

Some preliminary results are collected in Paragraph 6.1, their proofs are postponed to Paragraph 6.3. The proofs of the main results are provided in Paragraph 6.2.

6.1 Preliminary results

The first lemma provides some technical tools necessary to show Theorem 2(ii).

Lemma 1. *Let $\varphi : \mathbb{R}^d \times \mathbb{R}_+ \times S^{d-1} \to \mathbb{R}$ be the function defined by*

$$\varphi(x, r, v) = r^2 \left[1 + \frac{\langle x - rv,\, v\rangle}{\|x - rv\|} \right].$$

Then for all $v \in S^{d-1}$, $\varphi(\cdot, \cdot, v)$ is nonnegative and we have that

$$\forall x \in \mathbb{R}^d,\ \forall r \leq \|x\|,\ \varphi(x, r, v) \leq 2r^2 \quad \text{and} \quad \forall r > \|x\|,\ \varphi(x, r, v) \leq \|x\|^2.$$

In particular, $\varphi(x, r, v) \leq 2\|x\|^2$ for every $(x, r, v) \in \mathbb{R}^d \times \mathbb{R}_+ \times S^{d-1}$.

The next lemma is the first step to prove Theorem 2(i).

Lemma 2. *Let $u \in S^{d-1}$. If $\mathbb{E}\|X\| < \infty$ then, for all $v \in \mathbb{R}^d$,*

$$\|q(\alpha u)\| \left\langle \alpha u - \frac{q(\alpha u)}{\|q(\alpha u)\|},\, v \right\rangle \to -\mathbb{E}\langle X - \langle X,\, u\rangle u,\, v\rangle \quad \text{as} \quad \alpha \uparrow 1.$$

Lemma 3 below is a result which is similar to Lemma 2.

Lemma 3. *Let $u \in S^{d-1}$. If $\mathbb{E}\|X\|^2 < \infty$ then*

$$\|q(\alpha u)\|^2 \left\langle \alpha u - \frac{q(\alpha u)}{\|q(\alpha u)\|},\, \frac{q(\alpha u)}{\|q(\alpha u)\|} \right\rangle \to -\frac{1}{2}\mathbb{E}\|X - \langle X,\, u\rangle u\|^2 \quad \text{as} \quad \alpha \uparrow 1.$$

Lemma 4 is the first step to prove Theorem 4. It is essentially a refinement of Lemma 2.

Lemma 4. *Let $u \in S^{d-1}$. If $\mathbb{E}\|X\|^2 < \infty$ then, for all $v \in \mathbb{R}^d$,*

$$\begin{aligned} & \|q(\alpha u)\| \left[\|q(\alpha u)\| \left\langle \alpha u - \frac{q(\alpha u)}{\|q(\alpha u)\|},\, v \right\rangle + \mathbb{E}\langle X - \langle X,\, u\rangle u,\, v\rangle \right] \\ \to \quad & \langle u,\, v\rangle \operatorname{Var}\langle X,\, u\rangle - \frac{1}{2}\langle u,\, v\rangle \mathbb{E}\|X - \langle X,\, u\rangle u\|^2 + \langle u,\, v\rangle \|\mathbb{E}(X - \langle X,\, u\rangle u)\|^2 - \operatorname{Cov}(\langle X,\, u\rangle, \langle X,\, v\rangle) \end{aligned}$$

as $\alpha \uparrow 1$.

Lemma 5 below is a refinement of Lemma 3. It is the second step to prove Theorem 4.

Lemma 5. *Let $u \in S^{d-1}$. If $\mathbb{E}\|X\|^3 < \infty$ then*

$$\begin{aligned} & \|q(\alpha u)\| \left(\|q(\alpha u)\|^2 \left\langle \alpha u - \frac{q(\alpha u)}{\|q(\alpha u)\|},\, \frac{q(\alpha u)}{\|q(\alpha u)\|} \right\rangle + \frac{1}{2}\mathbb{E}\|X - \langle X,\, u\rangle u\|^2 \right) \\ \to \quad & \|\mathbb{E}(X - \langle X,\, u\rangle u)\|^2 - \mathbb{E}(\langle X,\, u\rangle \|X - \langle X,\, u\rangle\|^2) \quad \text{as} \quad \alpha \uparrow 1. \end{aligned}$$

6.2 Proofs of the main results

Proof of Proposition 1. From Chaudhuri (1996), it is known that if $u \in B^d$ then problem (1) has a unique solution $q(u) \in \mathbb{R}^d$. To prove the converse part of this result, use equation (3) to get

$$\left\| \mathbb{E}\left(\frac{X - q(u)}{\|X - q(u)\|} \right) \right\| = \|u\|.$$

Introduce the coordinate representations $X = (X_1, \ldots, X_d)$ and $q(u) = (q_1(u), \ldots, q_d(u))$. The Cauchy-Schwarz inequality yields

$$\|u\|^2 = \left\| \mathbb{E}\left(\frac{X - q(u)}{\|X - q(u)\|} \right) \right\|^2 = \sum_{i=1}^{d} \left[\mathbb{E}\left(\frac{X_i - q_i(u)}{\|X - q(u)\|} \right) \right]^2 \leq \sum_{i=1}^{d} \mathbb{E}\left(\frac{(X_i - q_i(u))^2}{\|X - q(u)\|^2} \right) = 1.$$

Furthermore, equality holds if and only if for all $i \in \{1, \ldots, d\}$, there exists $\mu_i \in \mathbb{R}$ such that

$$\frac{X_i - q_i(u)}{\|X - q(u)\|} = \mu_i$$

almost surely. In particular, if $w = (\mu_1, \ldots, \mu_d)$, this entails $X \in D = q(u) + \mathbb{R}w$ almost surely, which cannot hold since the distribution of X is not concentrated in a single straight line in $\mathbb{R}^d$. It follows that necessarily $\|u\|^2 < 1$, which is the result. ■

Proof of Proposition 2. (i) Note that (3) implies that, for any linear isometry h of $\mathbb{R}^d$ and every $u \in B^d$,

$$h(u) + \mathbb{E}\left(\frac{h(X) - h \circ q(u)}{\|X - q(u)\|}\right) = 0.$$

Since h is a linear isometry, the random vectors X and $h(X)$ have the same distribution and the equality $\|X - q(u)\| = \|h(X) - h \circ q(u)\|$ holds almost surely. It follows that

$$h(u) + \mathbb{E}\left(\frac{X - h \circ q(u)}{\|X - h \circ q(u)\|}\right) = 0.$$

Since $h(u) \in B^d$, it follows that $h \circ q(u) = q \circ h(u)$, which completes the proof of the first statement.

(ii) To prove the second part of Proposition 2, start by noting that since X and $-X$ have the same distribution, it holds that $\mathbb{E}\left(X/\|X\|\right) = 0$. The case $u = 0$ is then obtained via (3). If $u \neq 0$, up to using the first part of the result with a suitable linear isometry, we shall assume without loss of generality that $u = (u_1, 0, \ldots, 0)$ for some constant $u_1 \in (0, 1)$. It is then enough to prove that there exists some constant $q_1(u) > 0$ such that $q(u) = (q_1(u), 0, \ldots, 0)$. To this end, let us remark that, on the one hand, if $v_1 \in \mathbb{R}$ and $v = v_1 w \in \mathbb{R}^d$ where $w = (1, 0, \ldots, 0)$ then

$$\forall j \in \{2, \ldots, d\},\ \mathbb{E}\left(\frac{X_j}{\|X - v_1 w\|}\right) = 0, \tag{5}$$

since, for every $j \in \{2, \ldots, d\}$, the random vectors X and $(X_1, \ldots, X_{j-1}, -X_j, X_{j+1}, \ldots, X_d)$ have the same distribution. On the other hand, the dominated convergence theorem entails that the function

$$v_1 \mapsto \mathbb{E}\left(\frac{X_1 - v_1}{\|X - v_1 w\|}\right)$$

is continuous, converges to 1 at $-\infty$, is equal to 0 at 0 and converges to -1 at $+\infty$. Thus, the intermediate value theorem yields that there exists some constant $q_1(u) > 0$ such that

$$u_1 + \mathbb{E}\left(\frac{X_1 - q_1(u)}{\|X - q_1(u) w\|}\right) = 0. \tag{6}$$

Consequently, collecting (5) and (6) yields

$$u + \mathbb{E}\left(\frac{X - q_1(u) w}{\|X - q_1(u) w\|}\right) = 0$$

and it only remains to apply (3) to finish the proof of the second statement.

(iii) To show the third statement, use the first result to obtain that the function $g : \|u\| \mapsto \|q(u)\|$ is indeed well-defined; since the geometric quantile function is continuous, so is g. Assume that g is not increasing: namely, there exist $u_1,\ u_2 \in B^d$ such that $\|u_1\| < \|u_2\|$ and $\|q(u_1)\| \geq \|q(u_2)\|$. Since $\|q(0)\| = 0$, it is a consequence of the intermediate value theorem that one may find $u,\ v \in B^d$ such that $\|u\| < \|v\|$ and $\|q(u)\| = \|q(v)\|$. Let h be an isometry such that $h(u/\|u\|) = h(v/\|v\|)$; then

$$\|q(h(u))\| = \|q(u)\| = \|q(v)\| = \|q(h(v))\| \quad \text{and} \quad \frac{q(h(u))}{\|q(h(u))\|} = \frac{h(u)}{\|h(u)\|} = \frac{h(v)}{\|h(v)\|} = \frac{q(h(v))}{\|q(h(v))\|}.$$

In other words, $q(h(u))$ and $q(h(v))$ have the same direction and magnitude, so that they are necessarily equal, which entails that $h(u) = h(v)$ because the geometric quantile function is one-to-one. This is a contradiction because $\|h(u)\| = \|u\| < \|v\| = \|h(v)\|$, and the third statement is proven.

(iv) Assume that $\|q(u)\|$ does not tend to infinity as $\|u\| \to 1$; since g is increasing, it tends to a finite positive limit r. In other words, $\|q(u)\| < r$ for every $u \in B^d$, which is a contradiction since the geometric quantile function maps B^d onto $\mathbb{R}^d$, and the proof is complete. ■

Proof of Theorem 1. (i) If the first statement were false, then one could find a sequence (v_n) contained in B^d such that $\|v_n\| \to 1$ and such that $(\|q(v_n)\|)$ does not tend to infinity. Up to extracting a subsequence, one can assume that $(\|q(v_n)\|)$ is bounded. Again, up to extraction, one can assume that (v_n) converges to some $v_\infty \in S^{d-1}$ and that $(q(v_n))$ converges to some $q_\infty \in \mathbb{R}^d$. Moreover, it is straightforward to show that for every u_1, u_2, q_1, $q_2 \in \mathbb{R}^d$

$$|\psi(u_1, q_1) - \psi(u_2, q_2)| \leq \{1 + \|u_2\|\} \|q_2 - q_1\| + \|q_1\| \|u_2 - u_1\|$$

so that the function ψ is continuous on $\mathbb{R}^d \times \mathbb{R}^d$. Recall then that the definition of $q(v_n)$ implies that for every $q \in \mathbb{R}^d$, $\psi(v_n, q(v_n)) \leq \psi(v_n, q)$ and let n tend to infinity to obtain

$$q_\infty = \arg\min_{q \in \mathbb{R}^d} \psi(v_\infty, q).$$

Because $v \in S^{d-1}$, this contradicts Proposition 1, and the proof of the first statement is complete: $\|q(v)\| \to \infty$ as $\|v\| \to 1$.

(ii) Pick a sequence (v_n) of elements of B^d converging to u and remark that from (3),

$$v_n + \mathbb{E}\left(\frac{X - q(v_n)}{\|X - q(v_n)\|}\right) = 0$$

for every integer n. Hence, for n large enough, the following equality holds:

$$v_n + \mathbb{E}\left(\left\|\frac{X}{\|q(v_n)\|} - \frac{q(v_n)}{\|q(v_n)\|}\right\|^{-1} \left[\frac{X}{\|q(v_n)\|} - \frac{q(v_n)}{\|q(v_n)\|}\right]\right) = 0. \tag{7}$$

Since the sequence $(q(v_n)/\|q(v_n)\|)$ is bounded it is enough to show that its only accumulation point is u. Let then u^* be an accumulation point of this sequence. Since $\|q(v_n)\| \to \infty$, we may let $n \to \infty$ in (7) and use the dominated convergence theorem to obtain $u - u^* = 0$, which completes the proof. ■

Proof of Theorem 2. (i) Let $(u, w_1, \ldots, w_{d-1})$ be an orthonormal basis of $\mathbb{R}^d$ and consider the following expansion :

$$\frac{q(\alpha u)}{\|q(\alpha u)\|} = b(\alpha)u + \sum_{k=1}^{d-1} \beta_k(\alpha) w_k \tag{8}$$

where $b(\alpha), \beta_1(\alpha), \ldots, \beta_{d-1}(\alpha)$ are real numbers. It immediately follows that

$$\frac{q(\alpha u)}{\|q(\alpha u)\|} - u - \frac{1}{\|q(\alpha u)\|}\{\mathbb{E}(X) - \langle \mathbb{E}(X), u\rangle u\} = (b(\alpha) - 1)u + \sum_{k=1}^{d-1} \frac{\|q(\alpha u)\|\beta_k(\alpha) - \mathbb{E}\langle X, w_k\rangle}{\|q(\alpha u)\|} w_k. \tag{9}$$

Lemma 2 implies that

$$\|q(\alpha u)\| \left\langle \alpha u - \frac{q(\alpha u)}{\|q(\alpha u)\|},\, w_k \right\rangle = -\|q(\alpha u)\|\beta_k(\alpha) \to -\mathbb{E}\langle X, w_k\rangle \quad \text{as} \quad \alpha \uparrow 1 \tag{10}$$

for all $k \in \{1, \ldots, d-1\}$. Besides, let us note that $q(\alpha u)/\|q(\alpha u)\| \in S^{d-1}$ entails

$$b^2(\alpha) + \sum_{k=1}^{d-1} \beta_k^2(\alpha) = 1. \tag{11}$$

Theorem 1 shows that $b(\alpha) \to 1$ as $\alpha \uparrow 1$ and thus (10) yields:

$$\|q(\alpha u)\|(1 - b(\alpha)) = \frac{1}{2}\|q(\alpha u)\|(1 - b^2(\alpha))(1 + \mathrm{o}(1)) = \frac{1}{2}\|q(\alpha u)\| \sum_{k=1}^{d-1} \beta_k^2(\alpha)(1 + \mathrm{o}(1)) \to 0 \quad \text{as} \quad \alpha \uparrow 1. \tag{12}$$

Collecting (9), (10) and (12), we obtain

$$\frac{q(\alpha u)}{\|q(\alpha u)\|} - u - \frac{1}{\|q(\alpha u)\|}\left\{\mathbb{E}(X) - \langle \mathbb{E}(X),\, u\rangle u\right\} = \mathrm{o}\left(\frac{1}{\|q(\alpha u)\|}\right) \quad \text{as} \quad \alpha \uparrow 1$$

which is the first result.

(ii) Recall (8) and use Lemma 2 to obtain

$$\|q(\alpha u)\| \left\langle \alpha u - \frac{q(\alpha u)}{\|q(\alpha u)\|},\, w_k \right\rangle \to -\mathbb{E}\langle X,\, w_k\rangle \quad \text{as} \quad \alpha \uparrow 1,$$

for all $k \in \{1, \ldots, d-1\}$, leading to

$$\|q(\alpha u)\|^2 \beta_k^2(\alpha) \to \left[\mathbb{E}\langle X,\, w_k\rangle\right]^2 \quad \text{as} \quad \alpha \uparrow 1 \tag{13}$$

for all $k \in \{1, \ldots, d-1\}$. Recall (11) and use Lemma 3 to get

$$\|q(\alpha u)\|^2 \left[\alpha b(\alpha) - 1\right] \to -\frac{1}{2}\mathbb{E}\|X - \langle X,\, u\rangle u\|^2 \quad \text{as} \quad \alpha \uparrow 1. \tag{14}$$

Since $(u, w_1, \ldots, w_{d-1})$ is an orthonormal basis of $\mathbb{R}^d$, one has the identity

$$\|X - \langle X,\, u\rangle u\|^2 = \sum_{k=1}^{d-1} \langle X,\, w_k\rangle^2. \tag{15}$$

Collecting (13), (14) and (15) leads to

$$\|q(\alpha u)\|^2 \left[1 - \alpha b(\alpha) - \frac{1}{2}\sum_{k=1}^{d-1} \beta_k^2(\alpha)\right] \to \frac{1}{2}\sum_{k=1}^{d-1} \mathrm{Var}\langle X,\, w_k\rangle \quad \text{as} \quad \alpha \uparrow 1.$$

Therefore,

$$\|q(\alpha u)\|^2 \left[1 - \alpha b(\alpha) - \frac{1}{2}\left(1 - b^2(\alpha)\right)\right] \to \frac{1}{2}\sum_{k=1}^{d-1} \mathrm{Var}\langle X,\, w_k\rangle \quad \text{as} \quad \alpha \uparrow 1, \tag{16}$$

and easy calculations show that

$$1 - \alpha b(\alpha) - \frac{1}{2}\left(1 - b^2(\alpha)\right) = \frac{1}{2}\left[(1 - \alpha)(1 + \alpha) + (\alpha - b(\alpha))^2\right]. \tag{17}$$

Finally, in view of Lemma 2,

$$\|q(\alpha u)\| \left\langle \alpha u - \frac{q(\alpha u)}{\|q(\alpha u)\|},\, u \right\rangle \to 0 \quad \text{as} \quad \alpha \uparrow 1$$

which is equivalent to

$$\|q(\alpha u)\|^2 \left(\alpha - b(\alpha)\right)^2 \to 0 \quad \text{as} \quad \alpha \uparrow 1. \tag{18}$$

Collecting (16), (17) and (18), we obtain

$$\|q(\alpha u)\|^2 (1-\alpha) \to \frac{1}{2} \sum_{k=1}^{d-1} \operatorname{Var}\langle X,\, w_k\rangle \quad \text{as} \quad \alpha \uparrow 1.$$

Remarking that, for every orthonormal basis $(e_1, \ldots, e_d)$ of $\mathbb{R}^d$,

$$\sum_{k=1}^{d} \operatorname{Var}\langle X,\, e_k\rangle = \sum_{k=1}^{d} e_k' \Sigma e_k = \operatorname{tr}\Sigma \tag{19}$$

proves that

$$\|q(\alpha u)\|^2 (1-\alpha) \to \frac{1}{2}(\operatorname{tr}\Sigma - u'\Sigma u) \geq 0 \quad \text{as} \quad \alpha \uparrow 1.$$

Finally, note that if we had $\operatorname{tr}\Sigma - u'\Sigma u = 0$ then by (19) we would have that $\operatorname{Var}\langle X,\, w_k\rangle = 0$ for all $k \in \{1, \ldots, d-1\}$. Thus the projection of X onto the orthogonal complement of $\mathbb{R}u$ would be almost surely constant and X would be contained in a single straight line in $\mathbb{R}^d$, which is a contradiction. This completes the proof of Theorem 2. ■

Proof of Theorem 3. Note that

$$\sqrt{1-\alpha_n}\, \widehat{q}_n(\alpha_n u) \to \left[\frac{1}{2}\left(\operatorname{tr}\Sigma - u'\Sigma u\right)\right]^{1/2} u \tag{20}$$

almost surely as $n \to \infty$. Moreover, by Theorems 1 and 2

$$\sqrt{1-\alpha_n}\, q(\alpha_n u) = \sqrt{1-\alpha_n} \|q(\alpha_n u)\| \frac{q(\alpha_n u)}{\|q(\alpha_n u)\|} \to \left[\frac{1}{2}\left(\operatorname{tr}\Sigma - u'\Sigma u\right)\right]^{1/2} u \tag{21}$$

almost surely as $n \to \infty$. Combining (20) and (21) completes the proof. ■

Proof of Theorem 4. Consider the following representation:

$$\begin{aligned}
\sqrt{n(1-\alpha_n)}\left(\widehat{q}_n(\alpha_n u) - q(\alpha_n u)\right) &= T_{1,n} + T_{2,n} + T_{3,n} \\
\text{with} \quad T_{1,n} &= \sqrt{n}\left(\left[\frac{1}{2}\{\operatorname{tr}\widehat{\Sigma}_n - u'\widehat{\Sigma}_n u\}\right]^{1/2} - \left[\frac{1}{2}\{\operatorname{tr}\Sigma - u'\Sigma u\}\right]^{1/2}\right) \frac{q(\alpha_n u)}{\|q(\alpha_n u)\|}, \\
T_{2,n} &= \sqrt{n}\left(\left[\frac{1}{2}\{\operatorname{tr}\Sigma - u'\Sigma u\}\right]^{1/2} - \sqrt{1-\alpha_n}\|q(\alpha_n u)\|\right) \frac{q(\alpha_n u)}{\|q(\alpha_n u)\|} \\
\text{and} \quad T_{3,n} &= -\sqrt{n(1-\alpha_n)}\|\widehat{q}_n(\alpha_n u)\| \left(\frac{q(\alpha_n u)}{\|q(\alpha_n u)\|} - u\right).
\end{aligned}$$

We start by examining the convergence of $T_{1,n}$. Observe first that

$$\begin{aligned} T_{1,n} &= \sqrt{n}\frac{1}{\sqrt{2}}\frac{\{\operatorname{tr}\widehat{\Sigma}_n - u'\widehat{\Sigma}_n u\} - \{\operatorname{tr}\Sigma - u'\Sigma u\}}{\{\operatorname{tr}\widehat{\Sigma}_n - u'\widehat{\Sigma}_n u\}^{1/2} + \{\operatorname{tr}\Sigma - u'\Sigma u\}^{1/2}}\frac{q(\alpha_n u)}{\|q(\alpha_n u)\|} \\ &= \sqrt{n}\frac{\{\operatorname{tr}\widehat{\Sigma}_n - u'\widehat{\Sigma}_n u\} - \{\operatorname{tr}\Sigma - u'\Sigma u\}}{2\sqrt{2}\{\operatorname{tr}\Sigma - u'\Sigma u\}^{1/2}} u(1 + \mathrm{o}_{\mathbb{P}}(1)) \quad \text{as} \quad n \to \infty \end{aligned}$$

in view of Theorem 1(i) and from the consistency of $\widehat{\Sigma}_n$. Denote by M the Gaussian centred limit of $\sqrt{n}(\widehat{\Sigma}_n - \Sigma)$ (see *e.g.* Neudecker and Wesselman, 1990). Since the map $A \mapsto \operatorname{tr} A - u'Au$ is linear, it follows that

$$\sqrt{n}\frac{\{\operatorname{tr}\widehat{\Sigma}_n - u'\widehat{\Sigma}_n u\} - \{\operatorname{tr}\Sigma - u'\Sigma u\}}{2\sqrt{2}\{\operatorname{tr}\Sigma - u'\Sigma u\}^{1/2}} \xrightarrow{d} Y \quad \text{as} \quad n \to \infty$$

where Y is a centred Gaussian random variable. Now, clearly $Z := Yu$ is a Gaussian centred random vector and we have

$$T_{1,n} \xrightarrow{d} Z \quad \text{as} \quad n \to \infty. \tag{22}$$

The sequence $T_{2,n}$ is controlled in the following way: using Lemmas 4 and 5 and following the steps of the proof of Theorem 2(ii), we obtain

$$\|q(\alpha_n u)\|^2(1-\alpha_n) = \frac{1}{2}(\operatorname{tr}\Sigma - u'\Sigma u) + \mathrm{O}(\|q(\alpha_n u)\|^{-1}) = \frac{1}{2}(\operatorname{tr}\Sigma - u'\Sigma u) + \mathrm{O}(\sqrt{1-\alpha_n}) \quad \text{as} \quad n \to \infty.$$

As a consequence

$$\|T_{2,n}\| = \mathrm{O}\left(\sqrt{n(1-\alpha_n)}\right) = \mathrm{o}(1) \quad \text{as} \quad n \to \infty. \tag{23}$$

We conclude by controlling $T_{3,n}$. Theorem 2 entails

$$\begin{aligned} \|T_{3,n}\| &= \mathrm{O}_{\mathbb{P}}\left(\sqrt{n(1-\alpha_n)}\frac{\|\widehat{q}_n(\alpha_n u)\|}{\|q(\alpha_n u)\|}\right) \\ &= \mathrm{O}_{\mathbb{P}}\left(\sqrt{n(1-\alpha_n)}\left[\frac{\operatorname{tr}\widehat{\Sigma}_n - u'\widehat{\Sigma}_n u}{\operatorname{tr}\Sigma - u'\Sigma u}\right]^{1/2}\right) = \mathrm{O}_{\mathbb{P}}\left(\sqrt{n(1-\alpha_n)}\right) = \mathrm{o}_{\mathbb{P}}(1) \quad \text{as} \quad n \to \infty \end{aligned} \tag{24}$$

by the consistency of $\widehat{\Sigma}_n$. Combining (22), (23) and (24) completes the proof. ■

6.3 Proofs of the preliminary results

Proof of Lemma 1. The fact that φ is nonnegative and the inequality

$$\forall r \leq \|x\|, \; \varphi(x,r,v) \leq 2r^2 \tag{25}$$

are straightforward consequences of the Cauchy-Schwarz inequality. Furthermore, φ can be rewritten as

$$\varphi(x,r,v) = r^2\left[\frac{\|x - \langle x,\, v\rangle v\|^2}{\|x - rv\|\,[\|x - rv\| - \langle x - rv,\, v\rangle]}\right].$$

Let us now remark that, if $\|x\| < r$, then, by the Cauchy-Schwarz inequality, $\langle x - rv,\, v\rangle = \langle x,\, v\rangle - r < 0$ which makes it clear that

$$\varphi(x,r,v)\mathbb{1}_{\{\|x\|<r\}} \leq r^2\frac{\|x - \langle x,\, v\rangle v\|^2}{\|x - rv\|^2}\mathbb{1}_{\{\|x\|<r\}} =: \psi(x,r,v)\mathbb{1}_{\{\|x\|<r\}}. \tag{26}$$

Since $\|x-rv\|^2 = \|x\|^2 - 2r\langle x,\, v\rangle + r^2$, the function $\psi(x,\cdot,v)$ is differentiable on $(\|x\|,\,+\infty)$ and some easy computations yield

$$\frac{\partial\psi}{\partial r}(x,r,v) = 2r\left[\|x\|^2 - r\langle x,\, v\rangle\right]\frac{\|x-\langle x,\, v\rangle v\|^4}{\|x-rv\|^4}.$$

If $\langle x,\, v\rangle \le 0$ then $\psi(x,\cdot,v)$ is increasing on $(\|x\|,\,+\infty)$ and thus

$$\forall r > \|x\|,\;\; \psi(x,r,v) \le \lim_{r\to+\infty}\psi(x,r,v) = \|x-\langle x,\, v\rangle v\|^2 \le \|x\|^2. \tag{27}$$

Otherwise, if $\langle x,\, v\rangle > 0$ then $\psi(x,\cdot,v)$ reaches its global maximum over $[\|x\|,\,+\infty)$ at $\|x\|^2/\langle x,\, v\rangle$ and therefore,

$$\forall r > \|x\|,\;\; \psi(x,r,v) \le \psi\left(x,\,\frac{\|x\|^2}{\langle x,\, v\rangle},\,v\right) = \|x\|^2. \tag{28}$$

Collecting (26), (27) and (28) yields

$$\varphi(x,r,v)\mathbb{1}_{\{\|x\|<r\}} \le \|x\|^2\mathbb{1}_{\{\|x\|<r\}}. \tag{29}$$

Combining (25) and (29) shows that $\varphi(x,r,v) \le 2\|x\|^2$ for every $r>0$ and every $v\in S^{d-1}$ and completes the proof of the result. ∎

Proof of Lemma 2. Let $v\in\mathbb{R}^d$ and $W_\alpha(\cdot,v):\mathbb{R}^d\to\mathbb{R}$ be the function defined by

$$W_\alpha(x,v) = \left[\left\|\frac{x}{\|q(\alpha u)\|} - \frac{q(\alpha u)}{\|q(\alpha u)\|}\right\|^{-1} - 1\right]\left\langle\frac{x}{\|q(\alpha u)\|} - \frac{q(\alpha u)}{\|q(\alpha u)\|},\, v\right\rangle.$$

For n large enough, (3) entails

$$\left\langle \alpha u - \frac{q(\alpha u)}{\|q(\alpha u)\|},\, v\right\rangle + \mathbb{E}\left(W_\alpha(X,v)\right) + \frac{1}{\|q(\alpha u)\|}\mathbb{E}\langle X,\, v\rangle = 0. \tag{30}$$

It is therefore enough to show that

$$\|q(\alpha u)\|\mathbb{E}\left(W_\alpha(X,v)\right) \to -\langle u,\, v\rangle\mathbb{E}\langle X,\, u\rangle \;\text{ as }\; \alpha\uparrow 1. \tag{31}$$

Since, for every $x\in\mathbb{R}^d$,

$$\left\|\frac{x}{\|q(\alpha u)\|} - \frac{q(\alpha u)}{\|q(\alpha u)\|}\right\|^2 = 1 - \frac{2}{\|q(\alpha u)\|}\left\langle x,\,\frac{q(\alpha u)}{\|q(\alpha u)\|}\right\rangle + \frac{\|x\|^2}{\|q(\alpha u)\|^2}, \tag{32}$$

it follows from a Taylor expansion and Theorem 1 that

$$\|q(\alpha u)\|W_\alpha(X,v) \to -\langle u,\, v\rangle\langle X,\, u\rangle \;\text{ almost surely as }\; \alpha\uparrow 1. \tag{33}$$

Besides,

$$\begin{aligned}
&\left|\left\|\frac{x}{\|q(\alpha u)\|} - \frac{q(\alpha u)}{\|q(\alpha u)\|}\right\|^{-1} - 1\right| \\
=\;& \left\|\frac{x}{\|q(\alpha u)\|} - \frac{q(\alpha u)}{\|q(\alpha u)\|}\right\|^{-1}\left[1 + \left\|\frac{x}{\|q(\alpha u)\|} - \frac{q(\alpha u)}{\|q(\alpha u)\|}\right\|\right]^{-1}\left|\frac{2}{\|q(\alpha u)\|}\left\langle x,\,\frac{q(\alpha u)}{\|q(\alpha u)\|}\right\rangle - \frac{\|x\|^2}{\|q(\alpha u)\|^2}\right|,
\end{aligned}$$

and the Cauchy-Schwarz inequality yields

$$\left\| \frac{x}{\|q(\alpha u)\|} - \frac{q(\alpha u)}{\|q(\alpha u)\|} \right\|^{-1} \left\langle \frac{x}{\|q(\alpha u)\|} - \frac{q(\alpha u)}{\|q(\alpha u)\|},\, v \right\rangle \leq \|v\|.$$

Thus, using the triangular inequality and the Cauchy-Schwarz inequality, it follows that

$$|W_\alpha(x,v)| \leq \|v\| \left[1 + \left\| \frac{x}{\|q(\alpha u)\|} - \frac{q(\alpha u)}{\|q(\alpha u)\|} \right\| \right]^{-1} \frac{\|x\|}{\|q(\alpha u)\|} \left[2 + \frac{\|x\|}{\|q(\alpha u)\|} \right].$$

Consequently, one has

$$\|q(\alpha u)\|\, |W_\alpha(x,v)|\, \mathbb{1}_{\{\|x\| \leq \|q(\alpha u)\|\}} \leq 3\|v\| \|x\| \mathbb{1}_{\{\|x\| \leq \|q(\alpha u)\|\}}.$$

Furthermore, the reverse triangle inequality entails, for $x \in \mathbb{R}^d$ such that $\|x\| > \|q(\alpha u)\|$

$$\left[1 + \left\| \frac{x}{\|q(\alpha u)\|} - \frac{q(\alpha u)}{\|q(\alpha u)\|} \right\| \right]^{-1} \leq \frac{\|q(\alpha u)\|}{\|x\|},$$

and therefore,

$$\|q(\alpha u)\|\, |W_\alpha(x,v)|\, \mathbb{1}_{\{\|x\| > \|q(\alpha u)\|\}} \leq 3\|v\| \|x\| \mathbb{1}_{\{\|x\| > \|q(\alpha u)\|\}}.$$

Finally,

$$\|q(\alpha u)\|\, |W_\alpha(X,v)| \leq 3\|v\| \|X\|$$

so that the integrand in (31) is bounded from above by an integrable random variable. One can now recall (33) and apply the dominated convergence theorem to obtain (31). The proof is complete. ∎

Proof of Lemma 3. Let $Z_\alpha : \mathbb{R}^d \to \mathbb{R}$ be the function defined by

$$Z_\alpha(x) = 1 + \left\langle \frac{x - q(\alpha u)}{\|x - q(\alpha u)\|},\, \frac{q(\alpha u)}{\|q(\alpha u)\|} \right\rangle.$$

For n large enough, (3) yields

$$\left\langle \alpha u - \frac{q(\alpha u)}{\|q(\alpha u)\|},\, \frac{q(\alpha u)}{\|q(\alpha u)\|} \right\rangle + \mathbb{E}\left(Z_\alpha(X)\right) = 0 \tag{34}$$

and it thus remains to prove that

$$\|q(\alpha u)\|^2 \mathbb{E}\left(Z_\alpha(X)\right) \to \frac{1}{2}\mathbb{E}\|X - \langle X,\, u\rangle u\|^2 \quad \text{as} \quad \alpha \uparrow 1.$$

To this end, rewrite Z_α as

$$Z_\alpha(x) = 1 - \left\| \frac{x}{\|q(\alpha u)\|} - \frac{q(\alpha u)}{\|q(\alpha u)\|} \right\|^{-1} \left[1 - \frac{1}{\|q(\alpha u)\|} \left\langle x,\, \frac{q(\alpha u)}{\|q(\alpha u)\|} \right\rangle \right]. \tag{35}$$

It thus follows from equation (32), Theorem 1 and a Taylor expansion that

$$Z_\alpha(x) = \frac{1}{2\|q(\alpha u)\|^2} \left\langle x - \left\langle x,\, \frac{q(\alpha u)}{\|q(\alpha u)\|} \right\rangle \frac{q(\alpha u)}{\|q(\alpha u)\|},\, x \right\rangle (1 + \mathrm{o}(1))$$

for all $x \in \mathbb{R}^d$. Using Theorem 1 again, we then get

$$\|q(\alpha u)\|^2 Z_\alpha(X) \to \|X\|^2 - \langle X,\, u\rangle^2 = \|X - \langle X,\, u\rangle u\|^2 \ \text{ almost surely as } \ \alpha \uparrow 1. \tag{36}$$

To conclude the proof, let $\varphi : \mathbb{R}^d \times \mathbb{R}_+ \times S^{d-1} \to \mathbb{R}$ be the function defined by

$$\varphi(x, r, v) = r^2 \left[1 + \frac{\langle x - rv,\, v\rangle}{\|x - rv\|} \right].$$

Note that $\|q(\alpha u)\|^2 Z_\alpha(x) = \varphi(x, \|q(\alpha u)\|, q(\alpha u)/\|q(\alpha u)\|)$. By Lemma 1:

$$\|q(\alpha u)\|^2 Z_\alpha(X) = \varphi(X, \|q(\alpha u)\|, q(\alpha u)/\|q(\alpha u)\|) \leq 2\|X\|^2$$

and the right-hand side is an integrable random variable. Use then (36) and the dominated convergence theorem to complete the proof. ■

Proof of Lemma 4. Let $v \in \mathbb{R}^d$ and recall the notation

$$W_\alpha(x, v) = \left[\left\| \frac{x}{\|q(\alpha u)\|} - \frac{q(\alpha u)}{\|q(\alpha u)\|} \right\|^{-1} - 1 \right] \left\langle \frac{x}{\|q(\alpha u)\|} - \frac{q(\alpha u)}{\|q(\alpha u)\|},\, v \right\rangle$$

from the proof of Lemma 2. From (30) there, it is enough to show that

$$\begin{aligned} \|q(\alpha u)\| \mathbb{E}\left(\|q(\alpha u)\| W_\alpha(X, v) + \langle u,\, v\rangle \langle X,\, u\rangle \right) \to \frac{1}{2} \langle u,\, v\rangle \mathbb{E}\|X - \langle X,\, u\rangle u\|^2 - \langle u,\, v\rangle \operatorname{Var}\langle X,\, u\rangle \\ + \operatorname{Cov}(\langle X,\, u\rangle, \langle X,\, v\rangle) - \langle u,\, v\rangle \|\mathbb{E}(X - \langle X,\, u\rangle u)\|^2 \quad (37) \end{aligned}$$

as $\alpha \uparrow 1$. Use now (32) in the proof of Lemma 2, Theorem 2(i) and a Taylor expansion to obtain after some cumbersome computations that

$$\begin{aligned} & \|q(\alpha u)\| \left(\|q(\alpha u)\| W_\alpha(X, v) + \langle u,\, v\rangle \langle X,\, u\rangle \right) \\ = \; & \frac{1}{2} \langle u,\, v\rangle \|X - \langle X,\, u\rangle u\|^2 - \langle u,\, v\rangle \langle X,\, u\rangle \left(\langle X,\, u\rangle - \mathbb{E}\langle X,\, u\rangle \right) \\ + \; & \langle X,\, u\rangle \left(\langle X,\, v\rangle - \mathbb{E}\langle X,\, v\rangle \right) - \langle u,\, v\rangle \langle X,\, \mathbb{E}(X - \langle X,\, u\rangle u)\rangle + \sum_{j=0}^{2} \|X\|^j \varepsilon_j(\alpha, X, q(\alpha u)) \end{aligned}$$

with probability 1, where for all $j \in \{0, 1, 2\}$, $\varepsilon_j(\alpha, y, z) \to 0$ as $\max(1 - \alpha, \|y\|/\|z\|) \downarrow 0$. In particular

$$\begin{aligned} & \|q(\alpha u)\| \left(\|q(\alpha u)\| W_\alpha(X, v) + \langle u,\, v\rangle \langle X,\, u\rangle \right) \\ \to \; & \frac{1}{2} \langle u,\, v\rangle \|X - \langle X,\, u\rangle u\|^2 - \langle u,\, v\rangle \langle X,\, u\rangle \left(\langle X,\, u\rangle - \mathbb{E}\langle X,\, u\rangle \right) - \langle u,\, v\rangle \langle X,\, \mathbb{E}(X - \langle X,\, u\rangle u)\rangle \\ + \; & \langle X,\, u\rangle \left(\langle X,\, v\rangle - \mathbb{E}\langle X,\, v\rangle \right) \quad \text{almost surely as} \quad \alpha \uparrow 1. \qquad (38) \end{aligned}$$

The proof shall be complete provided we can apply the dominated convergence theorem to the left-hand side of (38). To this end, let $\delta > 0$ be such that

$$\alpha \in (1 - \delta, 1) \;\text{ and }\; \frac{\|X\|}{\|q(\alpha u)\|} < \delta \Rightarrow \max_{0 \leq j \leq 2} |\varepsilon_j(\alpha, X, q(\alpha u))| \leq 1.$$

Equality (38) thus entails for α close enough to 1:

$$\|q(\alpha u)\| \Big| \|q(\alpha u)\| W_\alpha(X, v) + \langle u,\, v\rangle \langle X,\, u\rangle \Big| \mathbb{1}_{\{X < \delta \|q(\alpha u)\|\}} \leq P_1(\|X\|) \mathbb{1}_{\{X < \delta \|q(\alpha u)\|\}}$$

where P_1 is a real polynomial of degree 2. Besides, it is a consequence of the definition of $W_\alpha(X, v)$ and the Cauchy-Schwarz inequality that

$$\|q(\alpha u)\| \Big| \|q(\alpha u)\| W_\alpha(X, v) + \langle u,\, v\rangle \langle X,\, u\rangle \Big| \mathbb{1}_{\{X \geq \delta \|q(\alpha u)\|\}} \leq \frac{2(1 + \delta)}{\delta^2} \|X\|^2 \mathbb{1}_{\{X \geq \delta \|q(\alpha u)\|\}}.$$

One can conclude that there exists a real polynomial P_2 of degree 2 such that

$$\|q(\alpha u)\|\Big|\|q(\alpha u)\|W_\alpha(X,v)+\langle u,\,v\rangle\langle X,\,u\rangle\Big|\leq P_2(\|X\|)$$

so that the integrand in (37) is bounded by an integrable random variable. Recall (38) and apply the dominated convergence theorem to complete the proof. ■

Proof of Lemma 5. The proof is similar to that of Lemma 4. Recall from the proof of Lemma 3 the notation

$$Z_\alpha(x)=1+\left\langle\frac{x-q(\alpha u)}{\|x-q(\alpha u)\|},\,\frac{q(\alpha u)}{\|q(\alpha u)\|}\right\rangle.$$

From (34) there, it is enough to show that

$$\|q(\alpha u)\|\mathbb{E}\left(\|q(\alpha u)\|^2 Z_\alpha(X)-\frac{1}{2}\mathbb{E}\|X-\langle X,\,u\rangle u\|^2\right)\to\mathbb{E}(\langle X,\,u\rangle\|X-\langle X,\,u\rangle\|^2)-\|\mathbb{E}(X-\langle X,\,u\rangle u)\|^2 \quad (39)$$

as $\alpha\uparrow 1$. We first use (32) in the proof of Lemma 2, equation (35) in the proof of Lemma 3, Theorem 2(i) and a Taylor expansion to obtain after some burdensome computations that

$$\begin{aligned}&\|q(\alpha u)\|\left(\|q(\alpha u)\|^2 Z_\alpha(X)-\frac{1}{2}\|X-\langle X,\,u\rangle u\|^2\right)\\&=\langle X,\,u\rangle\|X-\langle X,\,u\rangle\|^2-\langle X,\,\mathbb{E}(X-\langle X,\,u\rangle u)\rangle+\sum_{j=0}^{3}\|X\|^j\varepsilon_j(\alpha,X,q(\alpha u))\end{aligned} \quad (40)$$

with probability 1, where for $j\in\{0,1,2,3\}$, $\varepsilon_j(\alpha,y,z)\to 0$ as $\max(1-\alpha,\|y\|/\|z\|)\downarrow 0$. Especially

$$\|q(\alpha u)\|\left(\|q(\alpha u)\|^2 Z_\alpha(X)-\frac{1}{2}\|X-\langle X,\,u\rangle u\|^2\right)\to\langle X,\,u\rangle\|X-\langle X,\,u\rangle\|^2-\langle X,\,\mathbb{E}(X-\langle X,\,u\rangle u)\rangle \quad (41)$$

as $\alpha\uparrow 1$. Our aim is now to apply the dominated convergence theorem to the left-hand side of (39). To this end, pick $\delta>0$ such that

$$\alpha\in(1-\delta,1)\ \text{ and }\ \frac{\|X\|}{\|q(\alpha u)\|}<\delta\Rightarrow\max_{0\leq j\leq 3}|\varepsilon_j(\alpha,X,q(\alpha u))|\leq 1.$$

Equality (40) thus entails for α close enough to 1:

$$\|q(\alpha u)\|\left|\|q(\alpha u)\|^2 Z_\alpha(X)-\frac{1}{2}\|X-\langle X,\,u\rangle u\|^2\right|\mathbb{1}_{\{X<\delta\|q(\alpha u)\|\}}\leq P_1(\|X\|)\mathbb{1}_{\{X<\delta\|q(\alpha u)\|\}}$$

where P_1 is a real polynomial of degree 3. Moreover, the Cauchy-Schwarz inequality yields

$$\|q(\alpha u)\|\left|\|q(\alpha u)\|^2 Z_\alpha(X)-\frac{1}{2}\|X-\langle X,\,u\rangle u\|^2\right|\mathbb{1}_{\{X\geq\delta\|q(\alpha u)\|\}}\leq\frac{4+\delta^2}{2\delta^3}\|X\|^3\mathbb{1}_{\{X\geq\delta\|q(\alpha u)\|\}}.$$

Consequently, there exists a real polynomial P_2 of degree 3 such that

$$\|q(\alpha u)\|\left|\|q(\alpha u)\|^2 Z_\alpha(X)-\frac{1}{2}\|X-\langle X,\,u\rangle u\|^2\right|\leq P_2(\|X\|).$$

We conclude that the integrand in (39) is bounded by an integrable random variable. Recall (41) and apply the dominated convergence theorem to complete the proof. ■

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Distribution	Value of α	Error $e(\alpha)$	Error $E_n(\alpha)$		
			$n = 100$	$n = 200$	$n = 500$
Centred Gaussian $\mathcal{N}(0, 1/2, 1/2, 0)$	0.990	$2.55 \cdot 10^{-5}$	$1.29 \cdot 10^{-3}$	$6.50 \cdot 10^{-4}$	$2.93 \cdot 10^{-4}$
	0.995	$2.43 \cdot 10^{-5}$	$1.28 \cdot 10^{-3}$	$6.44 \cdot 10^{-4}$	$2.88 \cdot 10^{-4}$
	0.999	$5.75 \cdot 10^{-5}$	$1.30 \cdot 10^{-3}$	$6.70 \cdot 10^{-4}$	$3.16 \cdot 10^{-4}$
Centred Gaussian $\mathcal{N}(0, 1/2, 1/2, 1/6)$	0.990	$1.05 \cdot 10^{-4}$	$1.45 \cdot 10^{-3}$	$7.32 \cdot 10^{-4}$	$3.57 \cdot 10^{-4}$
	0.995	$4.34 \cdot 10^{-5}$	$1.37 \cdot 10^{-3}$	$6.65 \cdot 10^{-4}$	$2.89 \cdot 10^{-4}$
	0.999	$6.34 \cdot 10^{-5}$	$1.38 \cdot 10^{-3}$	$6.83 \cdot 10^{-4}$	$3.05 \cdot 10^{-4}$
Centred Gaussian $\mathcal{N}(0, 1/8, 3/4, 0)$	0.990	$6.05 \cdot 10^{-4}$	$1.79 \cdot 10^{-3}$	$1.17 \cdot 10^{-3}$	$8.23 \cdot 10^{-4}$
	0.995	$1.77 \cdot 10^{-4}$	$1.34 \cdot 10^{-3}$	$7.31 \cdot 10^{-4}$	$3.91 \cdot 10^{-4}$
	0.999	$5.96 \cdot 10^{-5}$	$1.20 \cdot 10^{-3}$	$6.02 \cdot 10^{-4}$	$2.70 \cdot 10^{-4}$
Double exponential $\mathcal{E}(2, 2, 2, 2)$	0.990	$9.30 \cdot 10^{-5}$	$2.69 \cdot 10^{-3}$	$1.47 \cdot 10^{-3}$	$6.37 \cdot 10^{-4}$
	0.995	$5.46 \cdot 10^{-5}$	$2.63 \cdot 10^{-3}$	$1.41 \cdot 10^{-3}$	$5.93 \cdot 10^{-4}$
	0.999	$6.32 \cdot 10^{-5}$	$2.63 \cdot 10^{-3}$	$1.39 \cdot 10^{-3}$	$5.97 \cdot 10^{-4}$
Double exponential $\mathcal{E}(2\sqrt{3}, 2\sqrt{3}, 2\sqrt{3/5}, 2\sqrt{3/5})$	0.990	$6.17 \cdot 10^{-4}$	$4.37 \cdot 10^{-3}$	$2.71 \cdot 10^{-3}$	$1.42 \cdot 10^{-3}$
	0.995	$2.24 \cdot 10^{-4}$	$3.89 \cdot 10^{-3}$	$2.26 \cdot 10^{-3}$	$9.96 \cdot 10^{-4}$
	0.999	$2.27 \cdot 10^{-4}$	$3.77 \cdot 10^{-3}$	$2.16 \cdot 10^{-3}$	$9.62 \cdot 10^{-4}$
Double exponential $\mathcal{E}(4, 2\sqrt{2/3}, 4, 2\sqrt{2/3})$	0.990	$1.64 \cdot 10^{-3}$	$4.13 \cdot 10^{-3}$	$2.81 \cdot 10^{-3}$	$2.16 \cdot 10^{-3}$
	0.995	$8.13 \cdot 10^{-4}$	$3.27 \cdot 10^{-3}$	$1.98 \cdot 10^{-3}$	$1.33 \cdot 10^{-3}$
	0.999	$6.62 \cdot 10^{-5}$	$2.40 \cdot 10^{-3}$	$1.23 \cdot 10^{-3}$	$5.62 \cdot 10^{-4}$

Table 1: Errors $e(\alpha)$ and $E_n(\alpha)$ in all cases.

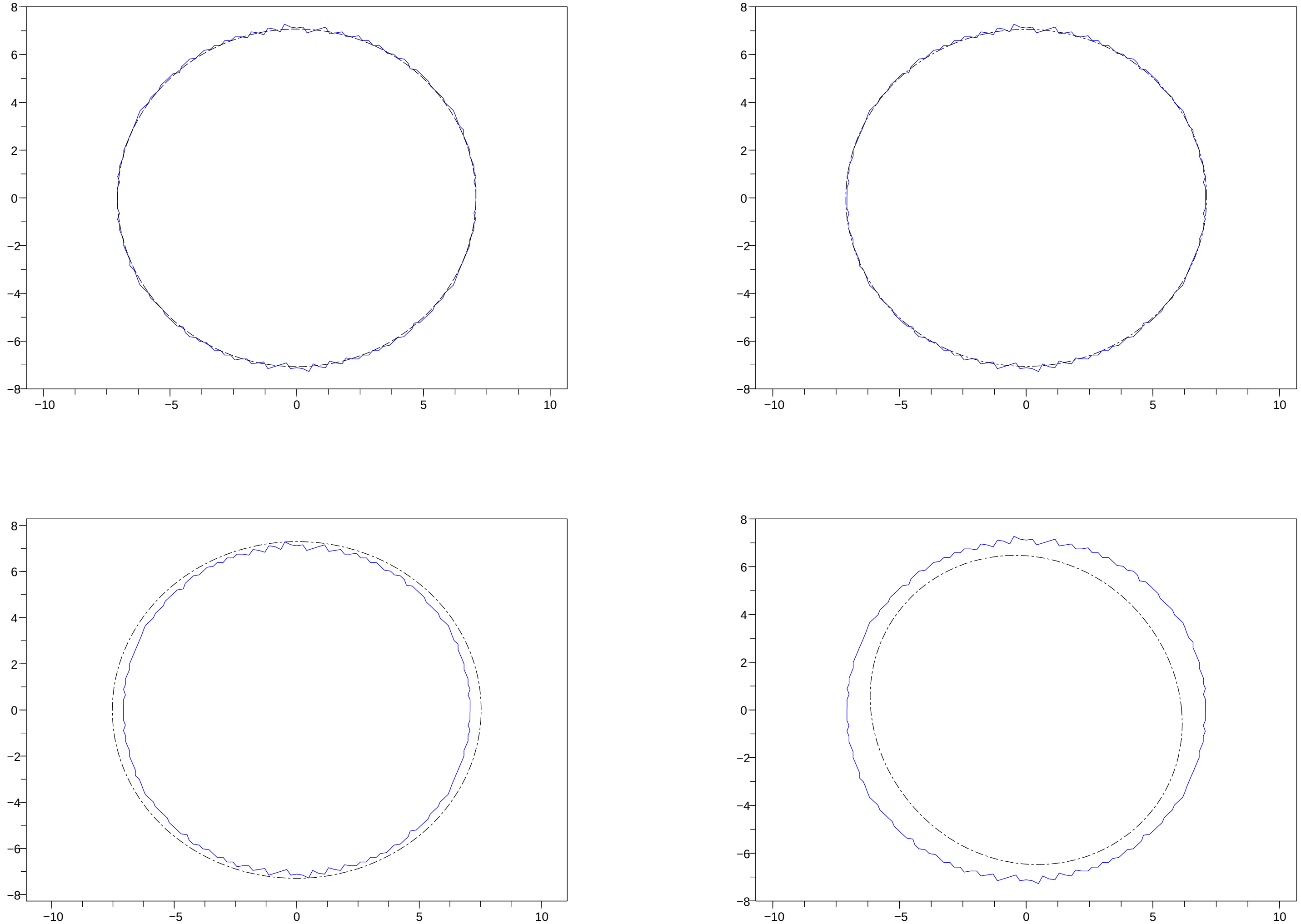

Figure 1: Case of the Gaussian distribution $\mathcal{N}(0, 1/2, 1/2, 0)$ for $\alpha = 0.995$. Top left: comparison between a numerical method and the use of the equivalent (4) for the computation of the iso-quantile curve, full line: numerical method, dashed line: asymptotic equivalent. Top right, bottom left and bottom right: best, median and worst estimates of the iso-quantile curve for $n = 200$, full line: numerical method, dashed-dotted line: estimator $\widehat{q}_n$.

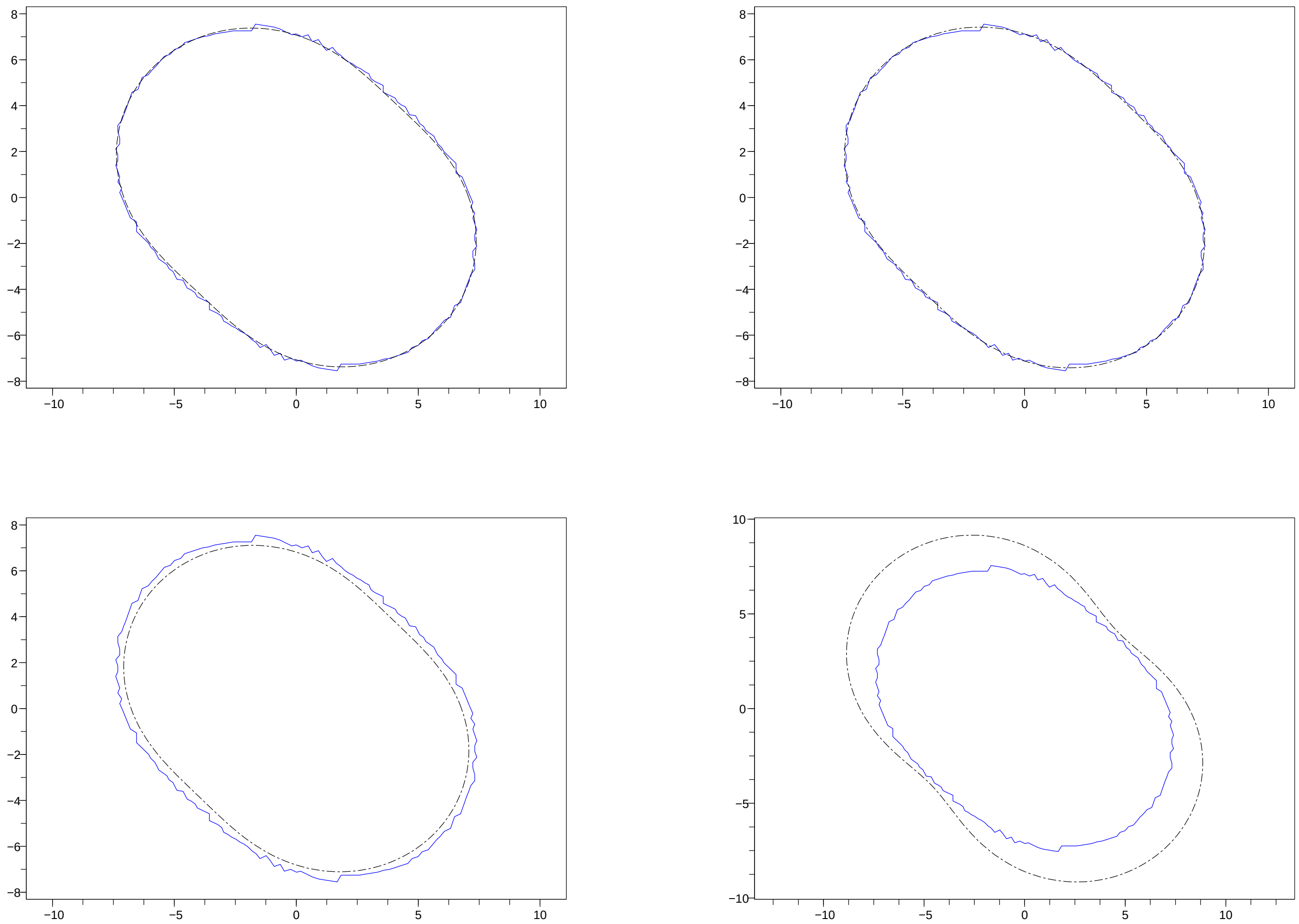

Figure 2: Case of the Gaussian distribution $\mathcal{N}(0, 1/2, 1/2, 1/6)$ for $\alpha = 0.995$. Top left: comparison between a numerical method and the use of the equivalent (4) for the computation of the iso-quantile curve, full line: numerical method, dashed line: asymptotic equivalent. Top right, bottom left and bottom right: best, median and worst estimates of the iso-quantile curve for $n = 200$, full line: numerical method, dashed-dotted line: estimator $\widehat{q}_n$.

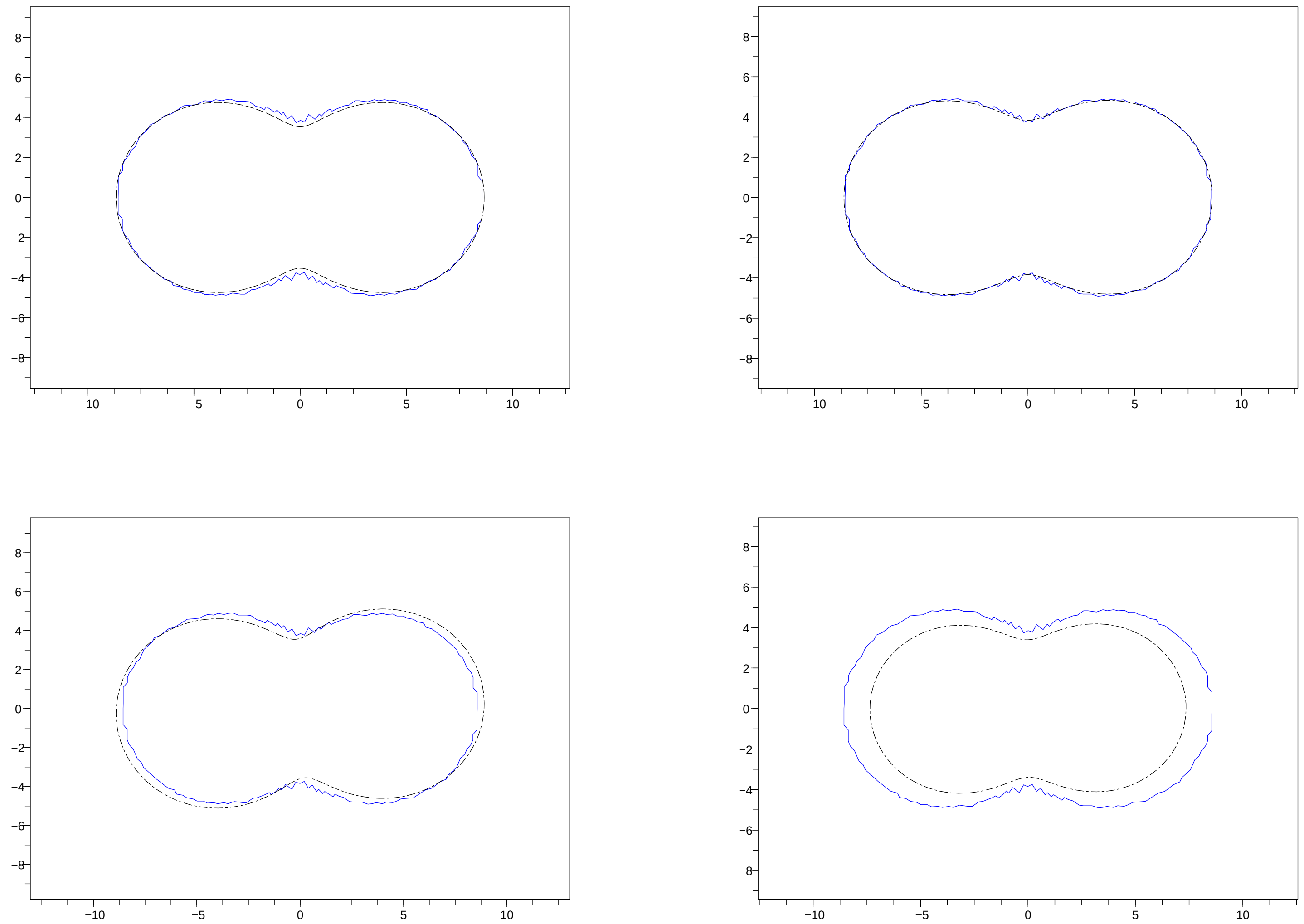

Figure 3: Case of the Gaussian distribution $\mathcal{N}(0, 1/8, 3/4, 0)$ for $\alpha = 0.995$. Top left: comparison between a numerical method and the use of the equivalent (4) for the computation of the iso-quantile curve, full line: numerical method, dashed line: asymptotic equivalent. Top right, bottom left and bottom right: best, median and worst estimates of the iso-quantile curve for $n = 200$, full line: numerical method, dashed-dotted line: estimator $\widehat{q}_n$.

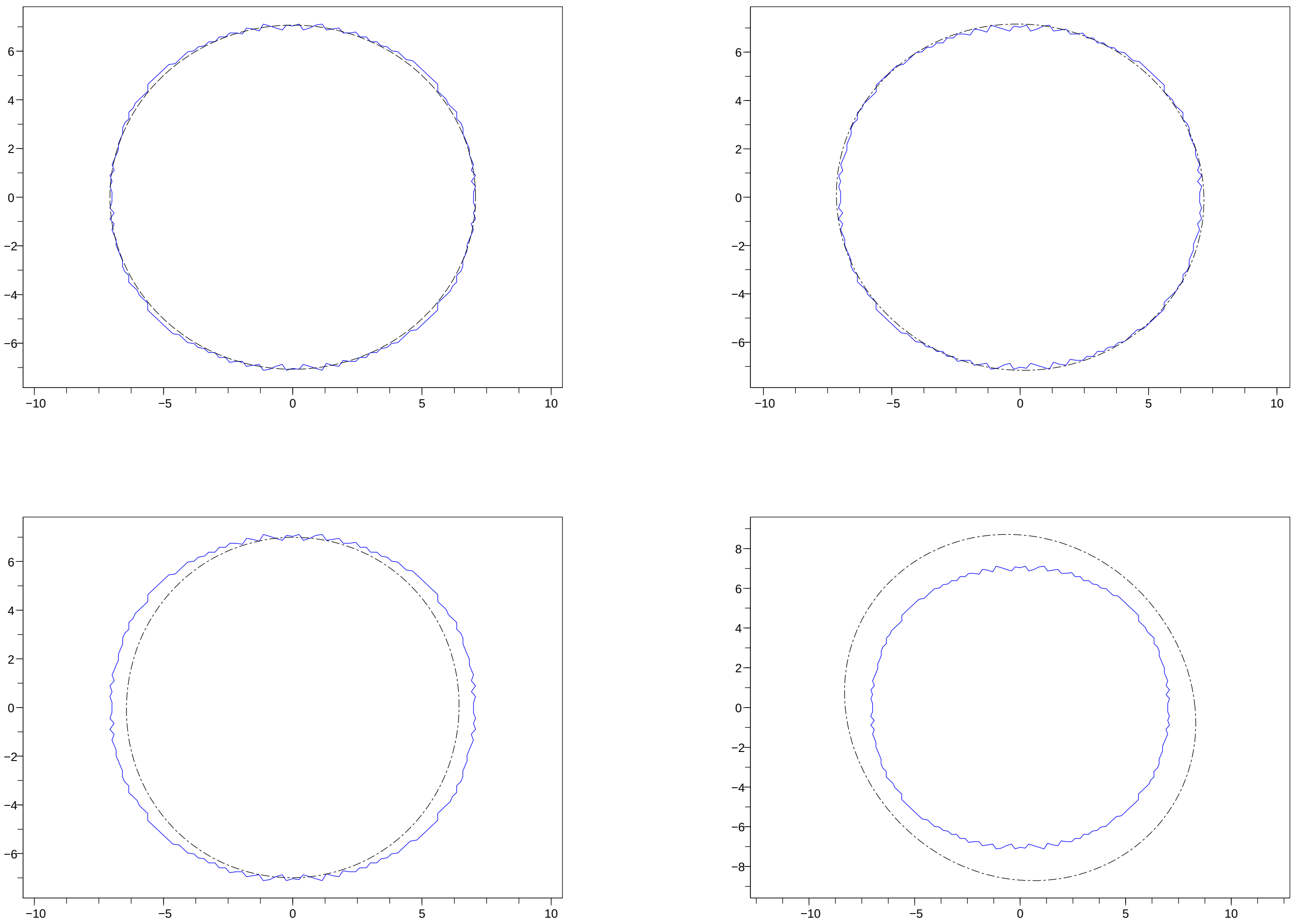

Figure 4: Case of the double exponential distribution $\mathcal{E}(2,2,2,2)$ for $\alpha = 0.995$. Top left: comparison between a numerical method and the use of the equivalent (4) for the computation of the iso-quantile curve, full line: numerical method, dashed line: asymptotic equivalent. Top right, bottom left and bottom right: best, median and worst estimates of the iso-quantile curve for $n = 200$, full line: numerical method, dashed-dotted line: estimator $\widehat{q}_n$.

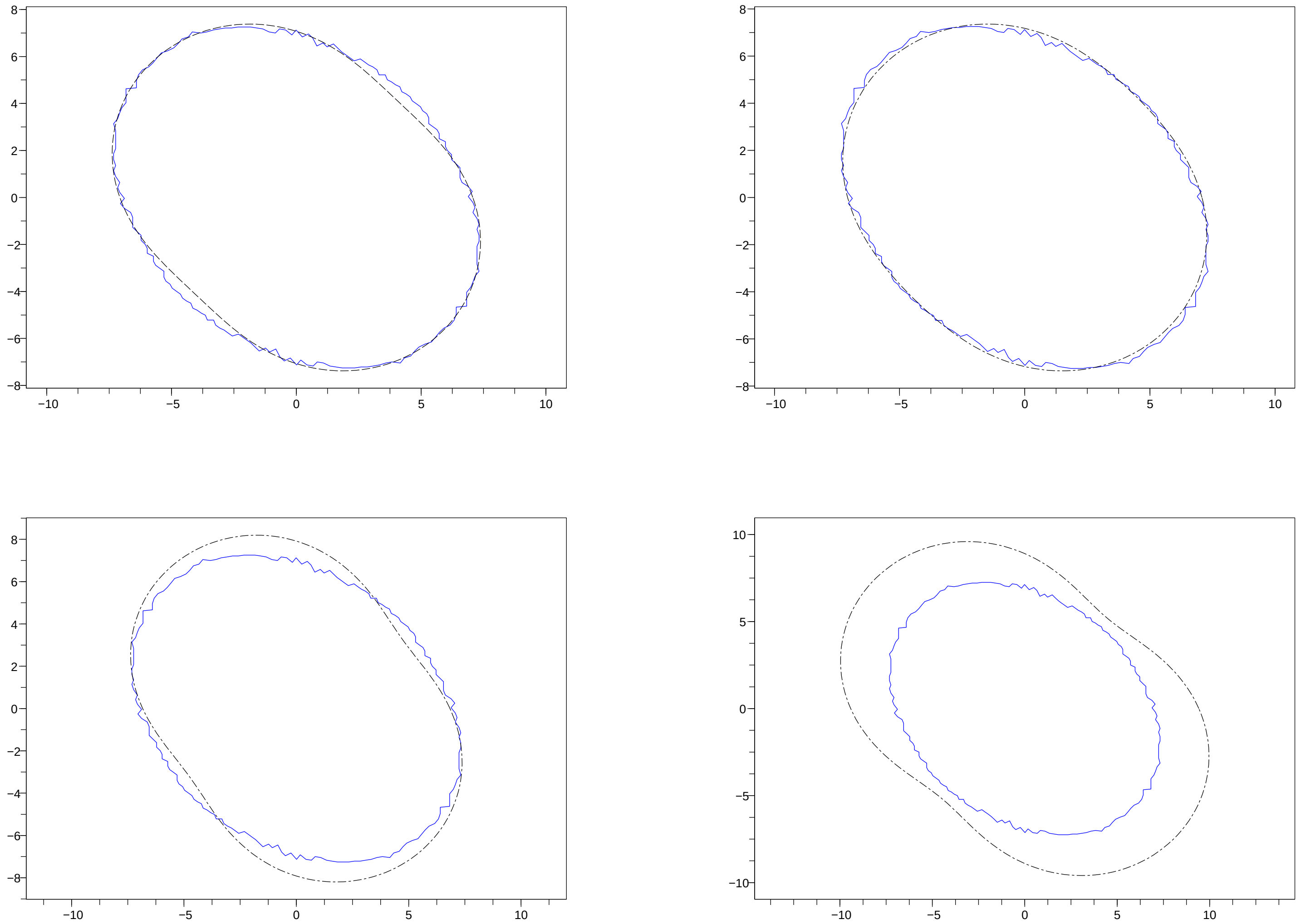

Figure 5: Case of the double exponential distribution $\mathcal{E}(2\sqrt{3}, 2\sqrt{3}, 2\sqrt{3/5}, 2\sqrt{3/5})$ for $\alpha = 0.995$. Top left: comparison between a numerical method and the use of the equivalent (4) for the computation of the iso-quantile curve, full line: numerical method, dashed line: asymptotic equivalent. Top right, bottom left and bottom right: best, median and worst estimates of the iso-quantile curve for $n = 200$, full line: numerical method, dashed-dotted line: estimator $\widehat{q}_n$.

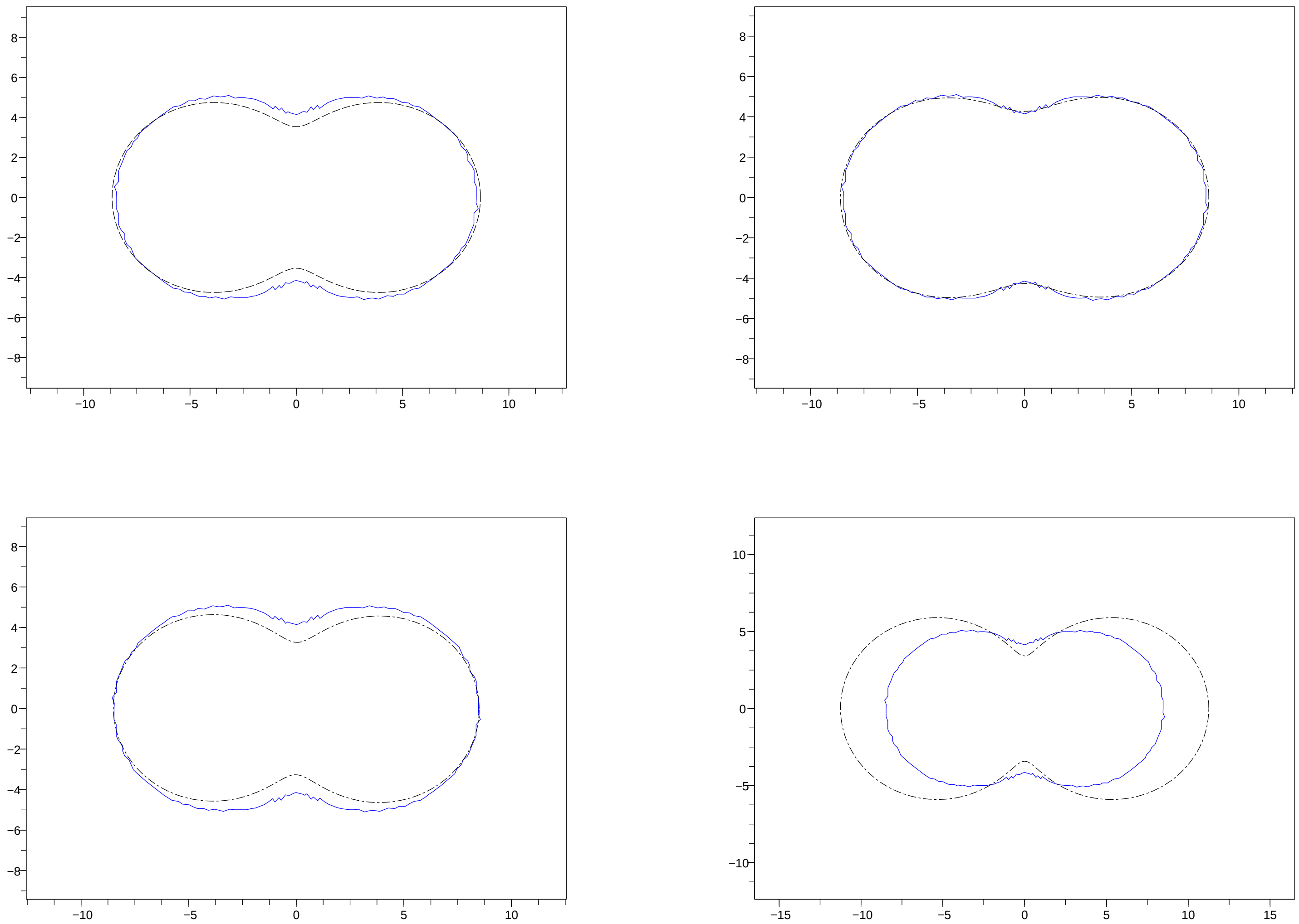

Figure 6: Case of the double exponential distribution $\mathcal{E}(4, 2\sqrt{2/3}, 4, 2\sqrt{2/3})$ for $\alpha = 0.995$. Top left: comparison between a numerical method and the use of the equivalent (4) for the computation of the iso-quantile curve, full line: numerical method, dashed line: asymptotic equivalent. Top right, bottom left and bottom right: best, median and worst estimates of the iso-quantile curve for $n = 200$, full line: numerical method, dashed-dotted line: estimator $\widehat{q}_n$.